Simple one-dimensional integral representations for two-loop self-energies: the master diagram

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Abstract

The scalar two-loop self-energy master diagram is studied in the case of arbitrary masses. Analytical results in terms of Lauricella- and Appell-functions are presented for the imaginary part. By using the dispersion relation a one-dimensional integral representation is derived. This representation uses only elementary functions and is thus well suited for a numerical calculation of the master diagram.

1 Introduction

The recent years have led to a considerable progress concerning the precision of measurements of some parameters of the standard model. Outstanding examples are the determination of the the Z-mass and -width [1]. Also a precise determination of the top-mass can be expected in the next years. As a consequence, very precise theoretical evaluations of the theory are more and more required, including calculations of two-loop corrections.

This paper focuses on the evaluation of the so-called scalar master diagram, which belongs to the class of two-loop self-energy diagrams. Analytical solutions, involving generalized hypergeometric functions, have been presented for all other scalar two-loop self-energy diagrams in the general mass case [2]-[4]. But no comparable results have so far been found for the master diagram, whereas for special mass configurations or in asymptotic parameter regions there exist many solutions [5]-[8]. In this paper we are concerned with the general mass case.

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After an introduction of our notations in sect. 2, we investigate in sect. 3 the imaginary part of the master diagram. Analytical results are presented for the two-particle cut contributions and three-particle cut contributions.

Concerning the full calculation of the master integral, it must be noted [9] that for two-loop integrals which contain a massive three-particle cut, the results in general cannot be expressed in terms of generalized polylogarithms with algebraic functions of the parameters as arguments. Thus, a promising approach to two-loop integrals uses integral representations which involve only elementary functions in the integrand and can therefore easily be evaluated numerically. For the master diagram a two-dimensional integral representation [10] has become famous, which involves only simple logarithms and square roots. This approach has been generalized to other two-loop self-energy integrals in ref. [11].

In sect. 4 we present a one-dimensional integral representation which only involves elementary functions. The standard algorithms for the numerical evaluation of one-dimensional integrals show very good convergence properties. Therefore the representation presented in this paper allows a very fast numerical calculation of the master diagram.

In sect. 5 we present some numerical comparisons of our results with those of [10], demonstrating the agreement of the results obtained with both methods.
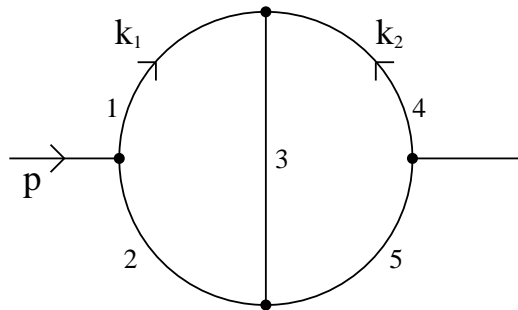
2   Notation and definitions

Figure 1: The master diagram T_{12345}.

The master integral (fig. 1) is in the general mass case determined by six independent parameters, i.e. five internal masses and the external squared momentum p^2. The name 'master diagram' has been introduced by Broadhurst and is due to the fact that a cancellation of propagators in this diagram generates all other fundamental two-loop self-energy diagrams. The integral $T_{12345}(p^2; m_1^2, m_2^2, m_3^2, m_4^2, m_5^2)$ is finite in four dimensions. We use the convention

$$T_{12345}(p^2; m_1^2, m_2^2, m_3^2, m_4^2, m_5^2) = -\frac{1}{\pi^4} \int d^4k_1 \, d^4k_2 \, \frac{1}{(k_1^2 - m_1^2)((k_1 - p)^2 - m_2^2)}$$
$$\times \frac{1}{((k_1 + k_2)^2 - m_3^2)(k_2^2 - m_4^2)((k_2 + p)^2 - m_5^2)} \, , \quad (1)$$

where all masses have an infinitely small negative imaginary part.

The integral T_{12345} has branch cuts along the positive real axis. The discontinuity across these cuts is related to the imaginary part,

$$\Delta T_{12345}(p^2; m_i^2) = 2i \operatorname{Im}\left(T_{12345}(p^2; m_i^2)\right),\tag{2}$$

where m_i^2 denotes the mass parameters. The integral can be calculated from the discontinuity with the dispersion relation

$$T_{12345}(p^2; m_i^2) = \frac{1}{2\pi i}\int\limits_{s_0}^{\infty} ds\,\frac{\Delta T_{12345}(s; m_i^2)}{s - p^2 - i\epsilon},\tag{3}$$

where s_0 denotes the lowest branch point of the function. According to the Cutkosky rules [12, 13], the discontinuity of the master diagram can be decomposed into four parts, each corresponding to a cut diagram (fig. 2),

$$\Delta T_{12345} = \Delta T_{12345}^{(2a)} + \Delta T_{12345}^{(2b)} + \Delta T_{12345}^{(3a)} + \Delta T_{12345}^{(3b)},\tag{4}$$

where $\Delta T_{12345}^{(2a)}$, $\Delta T_{12345}^{(2b)}$ and $\Delta T_{12345}^{(3a)}$, $\Delta T_{12345}^{(3b)}$ denote the two- and three-particle discontinuities, cf. fig. 2.
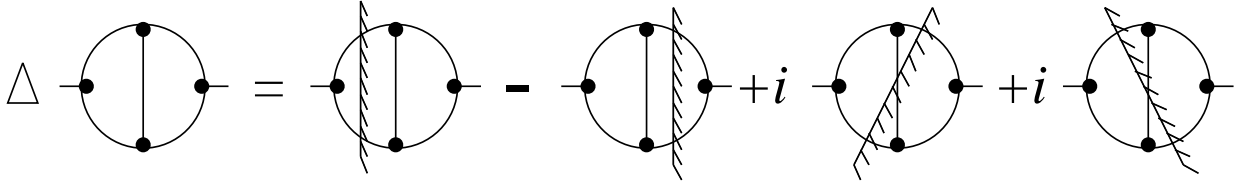

Figure 2: The contributions to the discontinuity of the master diagram.

Each cut gives a contribution to the dispersion integral, which will be denoted accordingly,

$$T_{12345}^{(2a)}(p^2; m_i^2) = \frac{1}{2\pi i}\int\limits_{s_0^{(2a)}}^{\infty} ds\,\frac{\Delta T_{12345}^{(2a)}(s; m_i^2)}{s - p^2 - i\epsilon}.\tag{5}$$

For each contribution, the lower limit of the integration is determined by the threshold which is associated with the cut, i.e. $s_0^{(2a)} = (m_1 + m_2)^2$, $s_0^{(2b)} = (m_4 + m_5)^2$, $s_0^{(3a)} = (m_2 + m_3 + m_4)^2$, $s_0^{(3b)} = (m_1 + m_3 + m_5)^2$.

3 Calculation of the imaginary part

3.1 Explicit results for the two-particle discontinuities

The two-particle cut contributions to the discontinuity are given according to the Cutkosky rules as

$$\Delta T_{12345}^{(2a)} = -\frac{1}{\pi^4}\int d^4k_1\, d^4k_2\, 2\pi\Theta(k_1^0)\delta(k_1^2 - m_1^2)2\pi\Theta(-k_1^0 + p^0)\delta((k_1 - p)^2 - m_2^2)$$

$$\times\frac{1}{((k_1 + k_2)^2 - m_3^2 - i\epsilon)(k_2^2 - m_4^2 - i\epsilon)((k_2 + p)^2 - m_5^2 - i\epsilon)},\tag{6}$$

$$\Delta T^{(2b)}_{12345} = \frac{1}{\pi^4} \int d^4k_1\, d^4k_2\, 2\pi\Theta(-k_2^0)\delta(k_2^2 - m_4^2)2\pi\Theta(k_2^0 + p^0)\delta((k_2 + p)^2 - m_5^2)$$

$$\times \frac{1}{(k_1^2 - m_1^2 + i\epsilon)((k_1 - p)^2 - m_2^2 + i\epsilon)((k_1 + k_2)^2 - m_3^2 + i\epsilon)}\,. \qquad (7)$$

In the evaluation of the expressions (6) and (7), the discontinuity of the one-loop self-energy in four dimensions,

$$\Delta B_0(p^2; m_1^2, m_2^2) = \frac{i}{\pi^2} \int d^4k\, 2\pi\Theta(k^0)\delta(k^2 - m_1^2)2\pi\Theta(p^0 - k^0)\delta\left((p - k)^2 - m_2^2\right)$$

$$= 2\pi i\, \frac{\sqrt{\lambda(p^2, m_1^2, m_2^2)}}{p^2} \Theta(p^2 - (m_1 + m_2)^2)\,, \qquad (8)$$

occurs, where the Källén function λ is defined by $\lambda(x, y, z) = (x - y - z)^2 - 4yz$. The expressions (6) and (7) yield

$$\Delta T^{(2a)}_{12345} = \frac{i}{\pi^2} \int d^4k_1\, 2\pi\Theta(k_1^0)\delta(k_1^2 - m_1^2)2\pi\Theta(p^0 - k_1^0)\delta((k_1 - p)^2 - m_2^2)$$

$$\times \left(C_0(p^2, k_1^2, (k_1 - p)^2; m_5^2, m_4^2, m_3^2)\right)^*$$

$$= \Delta B_0(p^2; m_1^2, m_2^2)\left(C_0(p^2, m_1^2, m_2^2; m_5^2, m_4^2, m_3^2)\right)^*\,, \qquad (9)$$

and equivalently,

$$\Delta T^{(2b)}_{12345} = \Delta B_0(p^2; m_4^2, m_5^2)C_0(p^2, m_4^2, m_5^2; m_2^2, m_1^2, m_3^2)\,, \qquad (10)$$

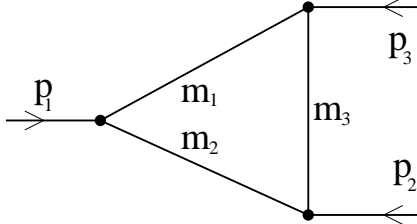where C_0 denotes the integral associated with the one-loop triangle diagram (fig. 3), whose

Figure 3: The triangle diagram.

analytical solution involves dilogarithms [14]. The two-particle discontinuities $\Delta T^{(2a)}_{12345}$ and $\Delta T^{(2b)}_{12345}$ are hence given by a product of the one-loop self-energy discontinuity ΔB_0 and the one-loop triangle integral C_0 or its complex conjugate C_0^*.

3.2   Integral representation for the three-particle discontinuity

For the three-particle discontinuities $\Delta T^{(3a)}_{12345}$ the Cutkosky rules yield

$$\Delta T^{(3a)}_{12345}(p^2; m_1^2, m_2^2, m_3^2, m_4^2, m_5^2)$$

$$= -\frac{i}{\pi^4} \int d^4k_1\, d^4k_2\, 2\pi\Theta(-k_1^0 + p^0)\delta((k_1 - p)^2 - m_2^2)2\pi\Theta(k_1^0 + k_2^0)\delta((k_1 + k_2)^2 - m_3^2)$$

$$\times 2\pi\Theta(-k_2^0)\delta(k_2^2 - m_4^2)\frac{1}{(k_1^2 - m_1^2 + i\epsilon)((k_2 + p)^2 - m_5^2 - i\epsilon)}\,. \qquad (11)$$

The contribution $\Delta T^{(3b)}_{12345}$ requires no separate evaluation. It is given by

$$\Delta T^{(3b)}_{12345}(p^2; m_1^2, m_2^2, m_3^2, m_4^2, m_5^2) = \Delta T^{(3a)}_{12345}(p^2; m_2^2, m_1^2, m_3^2, m_5^2, m_4^2). \tag{12}$$

With the dispersion representation of the propagator,

$$\frac{1}{k_1^2 - m_1^2 + i\epsilon} = \int_{-\infty}^{\infty} dt \, \frac{\delta(k_1^2 - t)}{t - m_1^2 + i\epsilon}, \tag{13}$$

the two momentum integrations in (11) can be performed separately and yield

$$\Delta T^{(3a)}_{12345}(p^2; m_i^2) = -\frac{1}{2\pi i} \int_{(m_3+m_4)^2}^{(\sqrt{p^2}-m_2)^2} dt \, \frac{\Delta B_0(p^2; t, m_2^2) \, \Delta C_0(t, m_2^2, p^2; m_4^2, m_3^2, m_5^2)}{t - m_1^2 + i\epsilon}. \tag{14}$$

The lower limit of the integration follows from the threshold of the ΔB_0-function (8), while the upper limit guarantees that $p^2 > (m_2 + m_3 + m_4)^2$, i.e. that p^2 is larger then the three-particle threshold. In expression (14), the function ΔC_0 denotes the discontinuity of the one-loop integral C_0 (fig. 3) with respect to the parameter p_1^2.

This discontinuity is the main ingredient in the dispersion representation of C_0, which will be used in sect. 4. The function C_0 can be calculated as

$$\begin{aligned}
C_0(p_1^2, p_2^2, p_3^2; m_1^2, m_2^2, m_3^2) &= \frac{1}{2\pi i} \int_{s_0}^{\infty} \frac{\Delta C_0(s, p_2^2, p_3^2; m_1^2, m_2^2, m_3^2)}{s - p_1^2 - i\epsilon} ds \\
&\quad + C_{0an}(p_1^2, p_2^2, p_3^2; m_1^2, m_2^2, m_3^2),
\end{aligned} \tag{15}$$

where the lower integration limit $s_0 = (m_1 + m_2)^2$ is the threshold belonging to p_1^2, ΔC_0 is the discontinuity associated with the normal threshold and C_{0an} is a contribution of an anomalous threshold. The discontinuity ΔC_0 is in general given by

$$\begin{aligned}
&\Delta C_0(s, p_2^2, p_3^2; m_1^2, m_2^2, m_3^2) \\
&= \frac{i}{\pi^2} \int d^4k \, 2\pi \Theta(k^0)\delta(k^2 - m_1^2) \, 2\pi\Theta(-k^0 - p_2^0 - p_3^0)\delta((k + p_2 + p_3)^2 - m_2^2) \\
&\quad \times \frac{1}{(k + p_3)^2 - m_3^2 - i\epsilon} \\
&= -2\pi i \, \frac{\log(a + b - i\epsilon) - \log(a - b - i\epsilon)}{\sqrt{(\Sigma p)^2 - s + i\epsilon} \sqrt{(\Delta p)^2 - s + i\epsilon}} \, \Theta(s - (m_1 + m_2)^2) \tag{16} \\
&= -2\pi i \, \frac{2\tanh^{-1}(b/a)}{\sqrt{(\Sigma p)^2 - s + i\epsilon} \sqrt{(\Delta p)^2 - s + i\epsilon}} \, \Theta(s - (m_1 + m_2)^2), \tag{17}
\end{aligned}$$

where

$$\begin{aligned}
&a\,(s, p_2^2, p_3^2; m_1^2, m_2^2, m_3^2) \\
&\quad = s\,(s + 2m_3^2 - p_2^2 - p_3^2 - m_1^2 - m_2^2) + (p_3^2 - p_2^2)(m_1^2 - m_2^2), \tag{18} \\
&b\,(s, p_2^2, p_3^2; m_1^2, m_2^2) = \sqrt{(\Sigma p)^2 - s + i\epsilon} \sqrt{(\Delta p)^2 - s + i\epsilon} \sqrt{\lambda(s, m_1^2, m_2^2)}, \tag{19} \\
&\Sigma p = |p_2| + |p_3|, \quad \Delta p = |p_2| - |p_3|,
\end{aligned}$$

and it is assumed that $p_2^2 > 0$ and $p_3^2 > 0$.

In order to produce a reliable numerical program for the master diagram, a careful discussion of the analytical properties of C_0 is required. The Landau equations show that a leading singularity and thus an anomalous threshold occurs in C_0 if

$$m_3^2 < m_{3thr}^2 = \frac{m_1 p_2^2 + m_2 p_3^2}{m_1 + m_2} - m_1 m_2 . \tag{20}$$

The branch point of this anomalous threshold is given by

$$s_1 = \frac{p_2^2 + p_3^2 + m_1^2 + m_2^2 - m_3^2}{2} - \frac{(p_2^2 - m_2^2)(p_3^2 - m_1^2) + l\,(p_2^2, m_2^2, m_3^2)\, l\,(p_3^2, m_1^2, m_3^2)}{2\,m_3^2} , \tag{21}$$

$$\text{with} \quad l\,(p_i^2, m_j^2, m_k^2) := \sqrt{p_i^2 - (m_j - m_k)^2 + i\epsilon}\,\sqrt{p_i^2 - (m_j + m_k)^2 + i\epsilon} .$$

One can understand the effect of the anomalous threshold by an inspection of the behavior of the branch point s_1 as a function of m_3^2, taking into account that this parameter has an infinitely small imaginary part. In terms of the dispersion representation (15), in which C_{0an} vanishes for $m_3^2 > m_{3thr}^2$, the threshold s_1 is a singularity of the logarithm in ΔC_0 as function of s. At $m_3^2 = m_{3thr}^2$ this singularity is equal to s_0. For $m_3^2 < m_{3thr}^2$ the singularity s_1 crosses in some cases the integration contour, which consequently has to be deformed. The additional contribution C_{0an} is therefore given by an integration over the discontinuity of the logarithm [15],

$$C_{0an} = \frac{1}{2\pi i} \int_{s_0}^{s_1} \frac{(2\pi i)^2}{\sqrt{\lambda(s, p_2^2, p_3^2)}} \frac{ds}{s - p_1^2} , \tag{22}$$

which evaluates to square roots and logarithms,

$$\int \frac{ds}{\sqrt{\lambda(s, p_2^2, p_3^2)}(s - p_1^2)} = \frac{1}{\sqrt{\lambda(p_1^2, p_2^2, p_3^2)}} \Big\{ \log(p_1^2 - s + i\epsilon)$$
$$- \log\big(- s(p_2^2 + p_3^2 - p_1^2) + (p_2^2 - p_3^2)^2 - p_1^2(p_2^2 + p_3^2)$$
$$+ \sqrt{(\Sigma p)^2 - s + i\epsilon}\sqrt{(\Delta p)^2 - s + i\epsilon}\sqrt{\lambda(p_1^2, p_2^2, p_3^2)} + i\epsilon \big) \Big\} . \tag{23}$$

From equation (22) it can be observed that for s_1 real and $s_1 > s_0$ no separate treatment of C_{0an} is required, but the discontinuity of the logarithm can be included as additional term to the logarithm in the discontinuity ΔC_0 (16).

Thus, one can distinguish the following six cases:

- *First case:* If $m_3^2 > m_{3thr}^2$, no anomalous threshold occurs.

- If $m_3^2 < m_{3thr}^2$ the anomalous threshold has to be taken into account. The behavior of its branch point s_1 at m_{3thr}^2 depends on the value of $\lambda(p_2^2, m_2^2, m_{3thr}^2)$.

  - If $\lambda(p_2^2, m_2^2, m_{3thr}^2) < 0$, which is equivalent to $\lambda(p_3^2, m_1^2, m_{3thr}^2) < 0$, the anomalous threshold deforms the integration contour. Three subcases may occur:
    * *Second case:* s_1 is real and $s_1 < s_0$.
    * *Third case:* s_1 takes on complex values.
    * *Fourth case:* s_1 is real and $s_1 > s_0$.

– If $\lambda(p_2^2, m_2^2, m_{3thr}^2) > 0$ the anomalous threshold deforms the integration contour only for complex valued s_1. Two subcases can be distinguished:

* *Fifth case:* s_1 is real and $s_1 > s_0$. Then the sheets of the logarithms in ΔC_0 change.
* *Sixth case:* s_1 takes on complex values.

For a numerical evaluation of the dispersion integral the following procedure for handling these cases can be applied. In the first case the discontinuity ΔC_0 can be calculated as given in (16). In the other cases the sheets of the logarithms in ΔC_0 differ from (16). In the second, third and sixth case the logarithms in (16) have to be replaced by

$$\log(a + b + i\epsilon) - \log(a - b - i\epsilon).$$

In the fourth case the additional discontinuity of the logarithm yields

$$\begin{cases} \log(a + b + i\epsilon) - \log(a - b - i\epsilon) - 2\pi i & \text{for} \quad s < s_1, \\ \log(a + b + i\epsilon) - \log(a - b - i\epsilon) & \text{for} \quad s > s_1. \end{cases}$$

In the fifth case the correct expressions are

$$\begin{cases} \log(a + b - i\epsilon) - \log(a - b - i\epsilon) & \text{for } s < s_1 \text{ and real } b, \\ \log(a + b) - \log(a - b) - 2\pi i & \text{for } s < s_1 \text{ and imaginary } b, \\ \log(a + b + i\epsilon) - \log(a - b - i\epsilon) & \text{for } s > s_1. \end{cases}$$

In the second, third and sixth case the contribution C_{0an} of the anomalous threshold, given by the expressions (22) and (23), has to be added. The argument of the second logarithm in (23) can get a negative imaginary part if s_1 has a nonvanishing imaginary part, i.e. in the third and sixth case. It has then to be replaced by the appropriate analytic continuation, which is given by

$$- \log \big(- s_1(p_2^2 + p_3^2 - p_1^2) + (p_2^2 - p_3^2)^2 - p_1^2(p_2^2 + p_3^2)$$
$$+ \sqrt{(\Sigma p)^2 - s_1 + i\epsilon}\sqrt{(\Delta p)^2 - s_1 + i\epsilon}\sqrt{\lambda(p_1^2, p_2^2, p_3^2)} \big) - 2\pi i. \tag{24}$$

The expressions (23) and (24) are valid for $p_1^2 > (|p_2| + |p_3|)^2$. An analytic continuation in terms of p_1^2 is easy, but is not required in the context of this paper.

We now return to the master diagram. Formula (14), which Broadhurst [5] took as starting point, is particularly useful, because the square root belonging to ΔB_0 cancels one of the square roots of ΔC_0, which results in

$$\Delta T_{12345}^{(3a)}(p^2; m_i^2) = \frac{2\pi i}{p^2} \int\limits_{(m_3 + m_4)^2}^{(\sqrt{p^2} - m_2)^2} dt \, \frac{1}{t - m_1^2 + i\epsilon} \, 2 \tanh^{-1} \frac{b\left(t, m_2^2, p^2, m_4^2, m_3^2\right)}{a\left(t, m_2^2, p^2, m_4^2, m_3^2, m_5^2\right)}, \tag{25}$$

with a and b defined as in (18) and (19). The argument of the $\tanh^{-1}$ is a rational function of t, p^2, m_2^2, m_3^2, m_4^2, m_5^2, and of $\sqrt{\lambda(t, p^2, m_2^2)\lambda(t, m_3^2, m_4^2)}$.

3.3 Representation of the three-particle discontinuities in terms of complete logarithmic elliptic integrals

A partial integration in (25) yields for the three-particle discontinuity

$$
\Delta T_{12345}^{(3a)}(p^2; m_i^2) = -\frac{4\pi i}{p^2} \int\limits_{(m_3+m_4)^2}^{(\sqrt{p^2}-m_2)^2} dt \log\left(\frac{t}{m_1^2} - 1 + i\epsilon\right)
$$
$$
\times \frac{\partial}{\partial t}\left(\tanh^{-1}\frac{b\,(t, m_2^2, p^2, m_4^2, m_3^2)}{a\,(t, m_2^2, p^2, m_4^2, m_3^2, m_5^2)}\right). \tag{26}
$$

Except for the logarithm, the form of (26) equals those of elliptic integrals. The zeros of the square root $\sqrt{\lambda(t, p^2, m_2^2)\lambda(t, m_3^2, m_4^2)}$ as a function of t are located at $(p \pm m_2)^2$ and $(m_3 \pm m_4)^2$. Therefore we introduce in the following the characteristic variables

$$
q_{\pm\pm} = (p \pm m_2)^2 - (m_3 \pm m_4)^2. \tag{27}
$$

In analogy to the case of the transformations of elliptic integrals to the Legendre normal form [16] one substitutes

$$
t = \frac{(m_3 + m_4)^2 q_{--} - (m_3 - m_4)^2 q_{-+} x^2}{q_{--} - q_{-+} x^2}, \tag{28}
$$

which leads to

$$
\Delta T_{12345}^{(3a)} = 2\pi i \int\limits_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\kappa^2 x^2)}} \left\{\log(a_0) + \log(1 - a_1 x^2) - \log(1 - a_2 x^2)\right\}
$$
$$
\times h^{(1)} \frac{r_4^{(1)} q_{-+}^4 x^8 + r_3^{(1)} q_{-+}^3 q_{--} x^6 + r_2^{(1)} q_{-+}^2 q_{--}^2 x^4 + r_1^{(1)} q_{-+} q_{--}^3 x^2 + r_0^{(1)} q_{--}^4}{\prod\limits_{i=1}^4 (q_{--} - c_i^{(1)} q_{-+} x^2)}
$$
$$
= 2\pi i \int\limits_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\kappa^2 x^2)}} \left\{\log(a_0) + \log(1 - a_1 x^2) - \log(1 - a_2 x^2)\right\}
$$
$$
\times h^{(1)} \left(h_0^{(1)} + \sum\limits_{i=1}^4 \frac{h_i^{(1)}}{1 - c_i^{(1)} \frac{q_{-+}}{q_{--}} x^2}\right), \tag{29}
$$

with

$$
a_0 = \frac{(m_3 + m_4)^2}{m_1^2}, \qquad a_1 = \frac{(m_3 - m_4)^2 - m_1^2}{(m_3 + m_4)^2 - m_1^2}\frac{q_{-+}}{q_{--}}, \qquad a_2 = \frac{q_{-+}}{q_{--}},
$$
$$
h^{(1)} = \frac{4\, m_3 m_4}{p^2 \sqrt{q_{++} q_{--}}\,(m_3 + m_4)^2\,[m_3(p^2 - m_4^2 - m_5^2) + m_4(m_2^2 - m_3^2 - m_5^2)]^2},
$$
$$
h_0^{(1)} = \frac{r_4^{(1)}}{c_1^{(1)} c_2^{(1)} c_3^{(1)} c_4^{(1)}},
$$

$$h_i^{(1)} = \frac{r_4^{(1)} + r_3^{(1)} c_i^{(1)} + r_2^{(1)} (c_i^{(1)})^2 + r_1^{(1)} (c_i^{(1)})^3 + r_0^{(1)} (c_i^{(1)})^4}{c_i^{(1)} \prod\limits_{j \neq i} (c_i^{(1)} - c_j^{(1)})} \quad \text{for } i = 1, \ldots, 4 \,,$$

$$c_1^{(1)} = 1 \,, \quad c_2^{(1)} = \frac{(m_3 - m_4)^2}{(m_3 + m_4)^2} \,,$$

$$c_{3/4}^{(1)} = \frac{(p^2 - m_4^2 - m_5^2)(m_2^2 - m_3^2 - m_5^2) - 4 m_3 m_4 m_5^2 \pm \sqrt{\lambda(m_5^2, p^2, m_4^2)\lambda(m_5^2, m_2^2, m_3^2)}}{(p^2 - m_4^2 - m_5^2)(m_2^2 - m_3^2 - m_5^2) + 4 m_3 m_4 m_5^2 \pm \sqrt{\lambda(m_5^2, p^2, m_4^2)\lambda(m_5^2, m_2^2, m_3^2)}} \,,$$

$$r_0^{(1)} = (m_3 + m_4) q_{-+} q_{++} (m_2^2 m_4 - m_3^2 m_4 - m_3 m_4^2 - m_3 m_5^2 - m_4 m_5^2 + m_3 p^2) \,,$$

$$\begin{aligned}
r_1^{(1)} = -2\Big\{ \big[&- m_4^2 (p^2 - m_2^2)^3 \\
&+ 2 m_4 (p^2 - m_2^2)(-m_2^4 m_3 + 2 m_2^2 m_3 m_4^2 + 3 m_2^2 m_4^3 \\
&\qquad + m_2^2 m_3 p^2 + 6 m_3 m_4^2 p^2 + 5 m_4^3 p^2) \\
&+ (m_3 + m_4)^2 (-6 m_2^4 m_3 m_4 + 6 m_2^2 m_3^3 m_4 - 2 m_3^5 m_4 + 4 m_2^4 m_4^2 \\
&\qquad + m_2^2 m_3^2 m_4^2 - 2 m_3^4 m_4^2 + m_2^2 m_4^4 - 2 m_3^2 m_4^4 \\
&\qquad - 2 m_3 m_4^5 + m_2^4 p^2 - 2 m_2^2 m_3^2 p^2 + m_3^4 p^2 \\
&\qquad + 4 m_2^2 m_3 m_4 p^2 + 2 m_2^2 m_4^2 p^2 + m_3^2 m_4^2 p^2 + 6 m_3 m_4^3 p^2 \\
&\qquad - 2 m_2^2 p^4 - 2 m_3^2 p^4 + 2 m_3 m_4 p^4 - 10 m_4^2 p^4 + p^6) \big] \\
&+ m_5^2 \big[- 2 m_3 m_4 (p^2 - m_2^2)^2 \\
&\qquad + (m_3 + m_4)^2 (-(p^2 - m_2^2)^2 + 2 m_2^2 m_3^2 - m_3^4 + 2 m_3^3 m_4 \\
&\qquad\qquad + 2 m_2^2 m_4^2 + 6 m_3^2 m_4^2 + 2 m_3 m_4^3 \\
&\qquad\qquad - m_4^4 + 2 m_3^2 p^2 + 2 m_4^2 p^2) \big] \Big\} \,,
\end{aligned}$$

$$r_2^{(1)} = 6\, m_3 m_4 (p^2 - m_2^2 + m_3^2 - m_4^2)(\lambda(p^2, m_4^2, m_5^2) - \lambda(m_2^2, m_3^2, m_5^2)) \,,$$

$$\begin{aligned}
r_3^{(1)} = -2\Big\{ \big[m_4(p^2 - m_2^2)(&-2 m_2^4 m_3 + m_2^4 m_4 + 4 m_2^2 m_3 m_4^2 - 6 m_2^2 m_4^3 + 2 m_2^2 m_3 p^2 \\
&- 2 m_2^2 m_4 p^2 + 12 m_3 m_4^2 p^2 - 10 m_4^3 p^2 + m_4 p^4) \\
+ (m_3 - m_4)^2 (&-6 m_2^4 m_3 m_4 + 6 m_2^2 m_3^3 m_4 - 2 m_3^5 m_4 + 10 m_4^2 p^4 - p^6 \\
&- 4 m_2^4 m_4^2 - m_2^2 m_3^2 m_4^2 + 2 m_3^4 m_4^2 - m_2^2 m_4^4 \\
&+ 2 m_3^2 m_4^4 - 2 m_3 m_4^5 - m_2^4 p^2 + 2 m_2^2 m_3^2 p^2 \\
&- m_3^4 p^2 + 4 m_2^2 m_3 m_4 p^2 - 2 m_2^2 m_4^2 p^2 - m_3^2 m_4^2 p^2 \\
&+ 6 m_3 m_4^3 p^2 + 2 m_2^2 p^4 + 2 m_3^2 p^4 + 2 m_3 m_4 p^4) \big] \\
+ m_5^2 \big[- 2 m_3 m_4 (p^2 - m_2^2)^2 & \\
+ (m_3 - m_4)^2 ((p^2 - m_2^2)^2 &- 2 m_2^2 m_3^2 + m_3^4 + 2 m_3^3 m_4 \\
&- 2 m_2^2 m_4^2 - 6 m_3^2 m_4^2 + 2 m_3 m_4^3 \\
&+ m_4^4 - 2 m_3^2 p^2 - 2 m_4^2 p^2) \big] \Big\} \,,
\end{aligned}$$

$$r_4^{(1)} = q_{--} q_{+-} (m_4 - m_3) \left[m_3(p^2 - m_4^2 - m_5^2) - m_4(m_2^2 - m_3^2 - m_5^2) \right] \,.$$

The integrals in (29) are either of the type of complete elliptic integrals or they have the

structure

$$\mathrm{LK}(a, \kappa) = \int_0^1 \frac{\log(1 - ax^2)dx}{\sqrt{(1-x^2)(1-\kappa^2 x^2)}} \tag{30}$$

$$\mathrm{L\Pi}(a, c, \kappa) = \int_0^1 \frac{\log(1 - ax^2)dx}{\sqrt{(1-x^2)(1-\kappa^2 x^2)}} \frac{1}{1 - cx^2} \,. \tag{31}$$

The analogy of these functions to the elliptic integrals is obvious. According to our knowledge they have not been introduced in the mathematical literature. We call them complete logarithmic elliptic integrals. For sake of completeness one can introduce a third function,

$$\mathrm{LE}(a, \kappa) = \int_0^1 \frac{(1-\kappa^2 x^2)\log(1 - ax^2)dx}{\sqrt{(1-x^2)(1-\kappa^2 x^2)}} \,. \tag{32}$$

Then all integrations of the type

$$\int_0^1 \frac{\log(1 - ax^2)dx}{\sqrt{(1-x^2)(1-\kappa^2 x^2)}} \, r(x) \,, \tag{33}$$

where $r(x)$ is an arbitrary rational function, can be expressed in terms of LK, LE, LΠ, complete elliptic integrals, logarithms and polylogarithms.

The functions LK, LE and LΠ are related to generalized hypergeometric functions of two and three variables, i.e. the Appell function $F_1(\alpha; \beta_1, \beta_2; \gamma; x, y)$ and the Lauricella function $F_D^{(3)}(\alpha; \beta_1, \beta_2, \beta_3; \gamma; x, y)$ [17],

$$\mathrm{LK}(a, \kappa) = -\frac{\pi}{2}\frac{\partial}{\partial \beta_2} \, \mathrm{F}_1(\frac{1}{2}; \frac{1}{2}, \beta_2; 1; \kappa^2, a)\Big|_{\beta_2 = 0} \,, \tag{34}$$

$$\mathrm{LE}(a, \kappa) = -\frac{\pi}{2}\frac{\partial}{\partial \beta_2} \, \mathrm{F}_1(\frac{1}{2}; -\frac{1}{2}, \beta_2; 1; \kappa^2, a)\Big|_{\beta_2 = 0} \,, \tag{35}$$

$$\mathrm{L\Pi}(a, c, \kappa) = -\frac{\pi}{2}\frac{\partial}{\partial \beta_3} \, \mathrm{F}_D(\frac{1}{2}; \frac{1}{2}, 1, \beta_3; 1; \kappa^2, c, a)\Big|_{\beta_3 = 0} \,. \tag{36}$$

The representation in terms of generalized hypergeometric functions helps to perform analytic continuations for LK, LK and LΠ. Series representations are given as

$$\mathrm{LK}(a, \kappa) = -\frac{\pi}{4}a \sum_{m,n=0}^{\infty} \frac{(\frac{1}{2})_m (\frac{3}{2})_{m+n}}{m!(m+n+1)!(n+1)}(\kappa^2)^m a^n \,, \tag{37}$$

$$\mathrm{LE}(a, \kappa) = -\frac{\pi}{4}a \sum_{m,n=0}^{\infty} \frac{(-\frac{1}{2})_m (\frac{3}{2})_{m+n}}{m!(m+n+1)!(n+1)}(\kappa^2)^m a^n \,, \tag{38}$$

$$\mathrm{L\Pi}(a, c, \kappa) = -\frac{\pi}{4}a \sum_{m,n,l=0}^{\infty} \frac{(\frac{1}{2})_m (\frac{3}{2})_{m+n+l}}{m!(m+n+l+1)!(l+1)}(\kappa^2)^m c^n a^l \,. \tag{39}$$

The region of convergence is given by $|\kappa^2| < 1$, $|a| < 1$, $|c| < 1$.

In terms of these functions one can finally express the result for $\Delta T^{(3a)}_{12345}$ as

$$\Delta T^{(3a)}_{12345}(p^2; m_i^2) = 2\pi i\, h^{(1)} \left\{ \log(a_0) \left(h_0^{(1)} \mathrm{K}(\kappa) + \sum_{i=1}^{4} h_i^{(1)}\, \Pi(\frac{q_{-+}}{q_{--}} c_i^{(1)}, \kappa) \right) \right.$$

$$+ \left(h_0^{(1)} \mathrm{LK}(a_1, \kappa) + \sum_{i=1}^{4} h_i^{(1)}\, \mathrm{L}\Pi(a_1, \frac{q_{-+}}{q_{--}} c_i^{(1)}, \kappa) \right)$$

$$\left. - \left(h_0^{(1)} \mathrm{LK}(a_2, \kappa) + \sum_{i=1}^{4} h_i^{(1)}\, \mathrm{L}\Pi(a_2, \frac{q_{-+}}{q_{--}} c_i^{(1)}, \kappa) \right) \right\}, \qquad (40)$$

a result which involves in an analogous way complete elliptic integrals and the complete logarithmic integrals introduced in (30), (31) and (32).

4 One-dimensional integral representations

In many cases dispersion representations have proven to be useful for the calculation of two-loop self-energy integrals [5, 6]. One can use this approach for further analytical calculations, or evaluate numerically the dispersion integral. In the case of the master diagram, the discontinuity in the general mass case is complicated to calculate, as shown in the last section. We will present two ways to simplify the evaluation of the dispersion integral. The first one is a realization of an idea of Broadhurst [5]. It simplifies the evaluation of the three-particle cut contributions by a partial integration in the dispersion integral. The second way relies on interchanging the integrations in both the two-particle cut contributions and the three-particle cut contributions of the master diagram and leads to a one-dimensional integral representation involving only elementary functions.

4.1 Evaluation of the three-particle cut contribution

Formula (9) expresses the two-particle cut contributions to the discontinuity in terms of well-known functions, though involving dilogarithms. However, for the three-particle cuts, equation (14) gives only an integral representation and expression (40) is a solution in terms of rather complicated series representations.

As Broadhurst has pointed out [5], it is useful to perform a partial integration in the dispersion integral for the three-particle cut contribution. One is then led to a suitable derivative, $\sigma'(p^2; m_i^2)$, of the discontinuity,

$$\sigma'(p^2; m_i^2) = \frac{1}{2\pi i} \frac{\partial}{\partial p^2} \left(p^2 \Delta T(p^2; m_i^2) \right) = \frac{1}{2\pi i} \left(\Delta T(p^2; m_i^2) + p^2 \frac{\partial \Delta T(p^2; m_i^2)}{\partial p^2} \right). \qquad (41)$$

The function T can then be calculated from σ' as

$$T(p^2; m_i^2) = -\frac{1}{p^2} \int_{s_0}^{\infty} ds\, \log(s - p^2 - i\epsilon)\, \sigma'(s; m_i^2). \qquad (42)$$

Equation (25) yields for $\sigma^{(3a)'}(p^2; m_i^2)$ of the three-particle cut contribution $T^{(3a)}_{12345}$

$$\sigma^{(3a)'}(p^2; m_i^2) = 2 \int_{(m_3+m_4)^2}^{(\sqrt{p^2}-m_2)^2} \frac{dt}{t - m_1^2} \frac{\partial}{\partial p^2} \left(\tanh^{-1} \frac{b(t, m_2^2, p^2, m_4^2, m_3^2)}{a(t, m_2^2, p^2, m_4^2, m_3^2, m_5^2)} \right). \qquad (43)$$

After a lengthy decomposition into partial fractions one finally arrives at a surprisingly simple result, involving only three Π-functions, i.e. complete elliptic integrals of the third kind,

$$
\begin{aligned}
\sigma^{(3a)'}(p^2;m_i^2) &= \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\kappa^2 x^2)}} h^{(2)} \sum_{i=1}^{3} \frac{r_1^{(2)} q_{--} q_{-+}^2 x^4 + r_0^{(2)} q_{--}^2 q_{-+} x^2}{\prod\limits_{i=1}^{3}(q_{--} - q_{-+} c_i^{(2)} x^2)} \\
&= \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\kappa^2 x^2)}} h^{(2)} \sum_{i=1}^{3} \frac{h_i^{(2)}}{1 - \frac{q_{-+}}{q_{--}} c_i^{(2)} x^2} \\
&= h^{(2)} \sum_{i=1}^{3} h_i^{(2)} \Pi(\frac{q_{-+}}{q_{--}} c_i^{(2)}, \kappa),
\end{aligned}
\tag{44}
$$

with variables,

$$
\kappa^2 = \frac{q_{+-} q_{-+}}{q_{++} q_{--}}, \qquad c_1^{(2)} = \frac{(m_3 - m_4)^2 - m_1^2}{(m_3 + m_4)^2 - m_1^2},
$$

$$
c_{2/3}^{(2)} = \frac{(p^2 - m_4^2 - m_5^2)(m_2^2 - m_3^2 - m_5^2) - 4 m_3 m_4 m_5^2 \pm \sqrt{\lambda(m_5^2, p^2, m_4^2)\lambda(m_5^2, m_2^2, m_3^2)}}{(p^2 - m_4^2 - m_5^2)(m_2^2 - m_3^2 - m_5^2) + 4 m_3 m_4 m_5^2 \pm \sqrt{\lambda(m_5^2, p^2, m_4^2)\lambda(m_5^2, m_2^2, m_3^2)}},
$$

$$
h^{(2)} = \frac{32 m_3^2 m_4^2}{\sqrt{q_{++} q_{--}} \left[m_3(p^2 - m_4^2 - m_5^2) + m_4(m_2^2 - m_3^2 - m_5^2) \right]^2 \left[(m_3 + m_4)^2 - m_1^2 \right]},
$$

$$
h_i^{(2)} = \frac{r_1^{(2)} + r_0^{(2)} c_i^{(2)}}{\prod\limits_{j \neq i}(c_i^{(2)} - c_j^{(2)})} \quad \text{for } i = 1, 2, 3,
$$

$$
\begin{aligned}
r_0^{(2)} &= (m_2^2 - m_3^2)^2 - 2 m_3 m_4 (m_2^2 - m_3^2) + m_4^2(m_2^2 + m_3^2) \\
&\quad - m_5^2(m_2^2 + (m_3 + m_4)^2) - p^2(m_2^2 + m_3^2 - m_5^2),
\end{aligned}
$$

$$
r_1^{(2)} = -r_0^{(2)}(m_4 \leftrightarrow -m_4).
$$

4.2 An efficient one-dimensional integral representation for the master diagram

The dispersion integral for the master diagram involves in the integrand dilogarithms for the two-particle cut contributions and complete elliptic integrals for the three-particle cut contributions (42), as presented in the previous section. In this section we use a different approach in order to derive a one-dimensional integral representation with elementary functions. This is achieved in the following way.

For an evaluation of the two-particle cut contributions it proves useful to introduce the dispersion representation of the function C_0 as given in equation (15). After an insertion of (9) and (15) into (5), the integrations in the dispersion integral can be interchanged and lead to

$$
\begin{aligned}
&T_{12345}^{(2a)}(p^2; m_1^2, m_2^2, m_3^2, m_4^2, m_5^2) \\
&= \frac{1}{2\pi i} \int_{s_0}^{\infty} \frac{ds}{s - p^2 - i\epsilon} \Delta T_{12345}^{(2a)}(s; m_i^2)
\end{aligned}
$$

$$= -\left(\frac{1}{2\pi i}\right)^2 \int\limits_{t_0}^{\infty} dt \int\limits_{s_0}^{\infty} ds \frac{(\Delta C_0(t))^* \, \Delta B_0(s)}{(s - p^2 - i\epsilon)(t - s + i\epsilon)}$$

$$+ \frac{1}{2\pi i} \int\limits_{s_0}^{\infty} ds \frac{\Delta B_0(s; m_1^2, m_2^2)}{s - p^2 - i\epsilon} \left(C_{0an}(s, m_1^2, m_2^2; m_5^2, m_4^2, m_3^2)\right)^*$$

$$= -\frac{1}{2\pi i} \int\limits_{(m_4+m_5)^2}^{\infty} dt \frac{(\Delta C_0(t, m_1^2, m_2^2; m_5^2, m_4^2, m_3^2))^*}{t - p^2 - i\epsilon} \left(B_0(p^2; m_1^2, m_2^2) - B_0(t; m_1^2, m_2^2)\right)$$

$$+ \frac{1}{2\pi i} \int\limits_{(m_1+m_2)^2}^{\infty} ds \frac{\Delta B_0(s; m_1^2, m_2^2)}{s - p^2 - i\epsilon} \left(C_{0an}(s, m_1^2, m_2^2; m_5^2, m_4^2, m_3^2)\right)^* . \tag{45}$$

Thus, one ends up with two one-dimensional integrals. The infinite and the constant parts of the two B_0-functions [14] in (45) cancel. In both integrals the integrands are composed of logarithms and square roots only, as can be seen from equations (16) and (23). The two integrals can therefore easily be evaluated numerically.

For $\Delta T_{12345}^{(2b)}$ the analogous calculation yields

$$T^{(2b)}(p^2; m_1^2, m_2^2, m_3^2, m_4^2, m_5^2)$$

$$= \frac{1}{2\pi i} \int\limits_{(m_1+m_2)^2}^{\infty} dt \frac{\Delta C_0(t, m_4^2, m_5^2; m_2^2, m_1^2, m_3^2)}{t - p^2 - i\epsilon} \left(B_0(p^2; m_4^2, m_5^2) - \left(B_0(t; m_4^2, m_5^2)\right)^*\right)$$

$$+ \frac{1}{2\pi i} \int\limits_{(m_4+m_5)^2}^{\infty} ds \frac{\Delta B_0(s; m_4^2, m_5^2)}{s - p^2 - i\epsilon} C_{0an}(s, m_4^2, m_5^2; m_2^2, m_1^2, m_3^2) . \tag{46}$$

The three-particle discontinuity (14) can be transformed with the partial integration (26), yielding

$$\Delta T_{12345}^{(3a)}(s; m_i^2) \;=\; 2\pi i \int\limits_{(m_3+m_4)^2}^{(\sqrt{s}-m_2)^2} dt \, \log\left(\frac{t}{m_1^2} - 1 + i\epsilon\right)$$

$$\times \frac{\sqrt{\lambda(s, t, m_2^2)} \, R(s, t, m_2^2, m_3^2, m_4^2, m_5^2)}{s} , \tag{47}$$

where $R(s)$ is a rational function of s. In the general mass case, i.e. if all masses are different, it has only first order poles. This form is well suited for evaluating the dispersion integral,

$$T_{12345}^{(3a)}(p^2; m_i^2) = \frac{1}{2\pi i} \int\limits_{s_0}^{\infty} ds \frac{\Delta T_{12345}^{(3a)}(s; m_i^2)}{s - p^2 - i\epsilon} , \tag{48}$$

because $R(s)$ can be decomposed into partial fractions,

$$R(s) \;=\; -\frac{\sqrt{\lambda(t, m_3^2, m_4^2)}}{t \, m_3^2} \frac{1}{(s - s_1)(s - s_2)} \left(\frac{r_1^{(3)} s + r_2^{(3)}}{(s - s_3)(s - s_4)} + r_3^{(3)} s + r_4^{(3)}\right)$$

$$= -\frac{\sqrt{\lambda(t, m_3^2, m_4^2)}}{t \, m_3^2} \sum_{i=1}^{4} \frac{f_i(t, m_2^2, m_3^2, m_4^2, m_5^2)}{s - s_i} . \tag{49}$$

The values of the $r_i^{(3)}$ and s_i will be presented below. An interchange of integrations,

$$\int\limits_{(m_2+m_3+m_4)^2}^{\infty} ds \int\limits_{(m_3+m_4)^2}^{(\sqrt{s}-m_2)^2} dt = \int\limits_{(m_3+m_4)^2}^{\infty} dt \int\limits_{(m_2+\sqrt{t})^2}^{\infty} ds \,, \tag{50}$$

leads to integrations of the type

$$\int\limits_{(\sqrt{t}+m_2)^2}^{\infty} \frac{ds}{s-s_i} \frac{\sqrt{\lambda(s,t,m_2^2)}}{s} = B_0(s_i;t,m_2^2) \,, \tag{51}$$

i.e. one-loop self-energy integrals. In the final expression all ultraviolet divergencies of these B_0-functions cancel. The result for the three-particle cut contribution of the master diagram is then

$$T^{(3a)}(p^2;m_i^2) = \int\limits_{(m_3+m_4)^2}^{\infty} dt \, \log\left(\frac{t}{m_1^2} - 1 + i\epsilon\right)$$

$$\times \frac{\sqrt{\lambda(t,m_4^2,m_3^2)}}{t\,m_3^2} \sum_{i=1}^{4} f_i \frac{B_0(p^2;t,m_2^2) - B_0(s_i;t,m_2^2)}{s_i - p^2 - i\epsilon} \,, \tag{52}$$

with

$$s_{1/2} = \frac{t + m_2^2 + m_4^2 + m_5^2 - m_3^2}{2} + \frac{(t-m_4^2)(m_5^2-m_2^2) \pm \sqrt{\lambda(t,m_3^2,m_4^2)\lambda(m_2^2,m_3^2,m_5^2)}}{2m_3^2}$$

$$s_{3/4} = (m_2 \pm \sqrt{t})^2 \,,$$

$$f_1 = \frac{s_1 r_1^{(3)} + r_2^{(3)}}{\prod\limits_{i=2}^{4}(s_1 - s_i)} + \frac{s_1 r_3^{(3)} + r_4^{(3)}}{s_1 - s_2} \,, \quad f_2 = f_1(s_1 \leftrightarrow s_2) \,,$$

$$f_3 = \frac{s_3 r_1^{(3)} + r_2^{(3)}}{\prod\limits_{i\neq 3}(s_3 - s_i)} \,, \quad f_4 = f_3(s_3 \leftrightarrow s_4) \,,$$

$$r_1^{(3)} = t(2m_2^2 - m_5^2 + m_3^2) - m_2^2(m_4^2 - m_3^2) \,,$$

$$r_2^{(3)} = (t - m_2^2)(t(m_5^2 - m_3^2) - m_2^2(m_4^2 - m_3^2)) \,,$$

$$r_3^{(3)} = \frac{m_3^2(m_3^2 - t - m_4^2)}{\lambda(t,m_4^2,m_3^2)} \,,$$

$$r_4^{(3)} = m_3^2$$

$$+ \frac{t(2m_4^2 m_3^2 - m_4^2 m_2^2 + m_5^2 m_4^2 + m_5^2 m_3^2) + m_4^2(m_2^2 - m_5^2)(m_4^2 - m_3^2) + m_5^2 m_3^2(m_4^2 - m_3^2)}{\lambda(t,m_4^2,m_3^2)} \,.$$

In order to obtain the second three-particle cut contribution $T^{(3b)}$, the masses have to be interchanged according to $m_1 \leftrightarrow m_2$ and $m_4 \leftrightarrow m_5$.

In some parameter regions one has to be careful when evaluating the B_0-functions in (52). The external momentum variable p^2 can be assumed to have an infinitely small positive imaginary part. Consequently, $B_0(p^2 + i\epsilon; t, m_2^2)$ can be evaluated in the usual way. The functions $B_0(s_{3/4}; t, m_2^2)$ are always evaluated below the threshold, and present

therefore no difficulty. However, care has to be taken in the evaluation of $B_0(s_{1/2}; t, m_2^2)$. A brief inspection of the asymptotic behavior of $s_{1/2}$ shows that the functions $B_0(s_{1/2}; t, m_2^2)$ have to be evaluated below the threshold for large values of m_3^2. In fact, no difficulty occurs for $m_3^2 > (m_2 + m_5)^2$. But s_1 and s_2 get an imaginary part for

$$(m_2 - m_5)^2 < m_3^2 < (m_2 + m_5)^2 \, .$$

In that case, the functions $B_0(s_{1/2}; m_1^2, m_2^2)$ have to be evaluated for complex values of the variables $s_{1/2}$, which is not too complicated. For $m_3^2 < (m_2 - m_5)^2$, the imaginary part of the variables $s_{1/2}$ vanishes again. Then one can in general evaluate the B_0-functions with the usual algorithm which assumes $B_0(s_{1/2} + i\epsilon; t, m_2^2)$. However, there exists a threshold t_{thr1},

$$
\begin{aligned}
t_{thr1} \;=\;\; & \frac{1}{2m_2^2}\Big(\lambda(m_3^2, m_2^2, m_5^2) + 2m_2^2(m_3^2 + m_4^2) \\
& + (m_5^2 - m_2^2 - m_3^2)\sqrt{\lambda(m_2^2, m_3^2, m_5^2) + 4m_2^2 m_4^2}\Big) ,
\end{aligned}
\tag{53}
$$

and for $t > t_{thr1}$ the function $B_0(s_2; t, m_2^2)$ has to be evaluated as $B_0(s_2 - i\epsilon; t, m_2^2) = (B_0(s_2 + i\epsilon; t, m_2^2))^*$.

Another complication occurs in that case, if $s_2(t)$ equals p^2 at a value $t > t_{thr1}$. Then one denominator in (52) vanishes at a point where the numerator takes on the value of the discontinuity of a B_0-function. This happens at $t = t_{thr2}$, if the following conditions apply

$$\lambda(p^2, m_4^2, m_5^2) > 0 \, , \tag{54}$$

$$
\begin{aligned}
t_{thr1} < t_{thr2} = & \frac{1}{2m_5^2}\Big(m_5^2(p^2 + m_2^2 + m_3^2 + m_4^2 - m_5^2) + (m_2^2 - m_3^2)(m_4^2 - p^2) \\
& + \sqrt{\lambda(m_2^2, m_3^2, m_5^2)\lambda(p^2, m_4^2, m_5^2)}\Big) ,
\end{aligned}
\tag{55}
$$

$$t_{thr2} > t_0 = (m_3 + m_4)^2 \, , \tag{56}$$

$$m_3^2 + m_5^2 - m_2^2 - \sqrt{\lambda(m_2^2, m_3^2, m_5^2)} > 0 \, . \tag{57}$$

The condition (54) is necessary for t_{thr2} to be real valued. The conditions (55) and (56) are obviously necessary for the complication in question. The condition (57) can be understood from the asymptotic behavior of s_2 for large values of t. Thus, if $m_3^2 < (m_2 - m_5)^2$ and the conditions (54-57) apply, the principal value and the residue of the integral have to be calculated.

The calculation of the residue involves the derivative of s_2 with respect to t,

$$\frac{ds_2}{dt} = \frac{1}{2\,m_3^2}\left(m_3^2 - m_2^2 + m_5^2 + (m_3^2 + m_4^2 - t)\sqrt{\frac{\lambda(m_2^2, m_3^2, m_5^2)}{\lambda(t, m_3^2, m_4^2)}}\right) . \tag{58}$$

The calculation then yields

$$\int_{t_0}^{\infty} dt \, f(t)\frac{B_0(p^2; t, m_2^2) - B_0(s_2(t); t, m_2^2)}{s_2 - p^2 - i\epsilon} \tag{59}$$

$$= \mathrm{P} \int\limits_{t_0}^{\infty} dt\, f(t) \frac{B_0(p^2; t, m_2^2) - B_0(s_2(t); t, m_2^2)}{s_2 - p^2} \tag{60}$$

$$+ i\pi f(t_{thr2}) \frac{\Delta B_0(p^2; t_{thr2}, m_2^2)}{(ds_2)/(dt)|_{t=t_{thr2}}} \,. \tag{61}$$

Some special mass cases have to be considered. Double poles occur in $R(s)$ if $m_2^2 = 0$, in which case $s_3 = s_4$, or if $\lambda(m_2^2, m_5^2, m_3^2) = 0$, in which case $s_1 = s_2$. In the case of $m_1^2 = 0$ the partial integration (26) is modified. For $T^{(3b)}$, these conditions amount to $m_1^2 = 0$, $\lambda(m_1^2, m_4^2, m_3^2) = 0$ or $m_2^2 = 0$. The decomposition into partial fractions leads then to modified results involving functions

$$\int\limits_{(\sqrt{t}+m)^2}^{\infty} \frac{ds}{(s - s_i - i\epsilon)^2} \frac{\sqrt{\lambda(s, t, m^2)}}{s} = \frac{\partial B_0(s_i; t, m^2)}{\partial s_i} \,. \tag{62}$$

These derivatives of the B_0-function are finite in four dimensions. Another modification and considerable simplification of the decomposition into partial fractions occurs if $m_3^2 = 0$.

In our evaluation of the three-particle cut contributions the symmetry of the master diagram with respect to an simultaneous interchange of the masses $m_1^2 \leftrightarrow m_4^2$ and $m_2^2 \leftrightarrow m_5^2$ is hidden. Broadhurst has evaluated the master diagram for many cases of physical interest [5]. Note that all these cases can be understood to belong to the special cases specified above, if one takes the symmetries of the master diagram into account.

5   Numerical comparisons

The formulae (45) and (52) provide an efficient algorithm for a numerical calculation of the master diagram in the general mass case. The results have been checked by a comparison with Kreimer's two-dimensional integral representation [10]. We implemented both algorithms with C++ and performed a test for more than 150000 different values of the parameters. This should be sufficient to check the parameter region.

For the one-dimensional integral representation an adaptive Gauss-Kronrod algorithm, implemented in the QUADPACK-routines [18] proved suitable. For the two-dimensional integral we did not succeed with a Gauss integration. But following ref. [11], we used the VEGAS-routine [19] which implements an adaptive Monte Carlo integration algorithm.

The following tables are intended to compare typical results of the two algorithms. We have chosen a parameter region where p^2 has a similar magnitude as the internal masses, where the integral representation is most useful to handle. Table 1 gives the results of the one-dimensional integral representation, table 2 of the two-dimensional integral representation. In each row the results for the real and the imaginary part, the reported error and the CPU-time are presented. To get a better estimate of the real errors and some impression about the convergence qualities of the algorithms, two runs have been performed for each parameter set. In case of the one-dimensional representation the required accuracy has been 10^{-4} for the first run and 5×10^{-6} for the second run. In the case of the two-dimensional representations we used 10^5 function evaluations for the first run and 5×10^5 function evaluations for the second run. The numerical results for the imaginary parts are

reported even in those regions, where p^2 is below all thresholds, because the magnitude of the imaginary part gives another estimate for the errors.

The precision of the one-dimensional integral representation can be increased further by introducing asymptotic formulae for the integrand at large values of the integration variable. With one-dimensional integral representations for other two-loop self-energy integrals [3, 4] we made the experience that simple asymptotic formulae increase the maximal precision by several orders of magnitude. We are going to implement this also for the algorithm presented in this paper. A first version of the program can be requested from the authors.

The programs have been run on a workstation DEC 3000 AXP. Note that the error reported by the VEGAS-routine gives the size of a standard deviation, while the error reported by the QUADPACK-routine is not so clear defined, but is in general highly overestimated. This can be observed from the tables by a comparison of the results with different precision.

p^2	A	B	C	D	E
0.1	<	$-2.87238e-01$	$-1.84514e-09$	$4.9e-05$	0.6
0.1	<	$-2.87238e-01$	$-1.57367e-10$	$2.5e-06$	0.8
0.5	<	$-2.94592e-01$	$-1.89266e-09$	$5.0e-05$	0.7
0.5	<	$-2.94592e-01$	$-1.61303e-10$	$2.5e-06$	0.8
1.0	<	$-3.04522e-01$	$-1.95521e-09$	$5.0e-05$	0.7
1.0	<	$-3.04521e-01$	$-1.66484e-10$	$2.5e-06$	0.8
5.0	<	$-4.52521e-01$	$-2.62945e-09$	$5.6e-05$	0.7
5.0	<	$-4.52520e-01$	$-2.22337e-10$	$2.8e-06$	0.8
10.0	>	$-4.88154e-01$	$-3.53218e-01$	$4.9e-05$	0.7
10.0	>	$-4.88153e-01$	$-3.53217e-01$	$2.3e-06$	0.9
50.0	>	$1.73902e-01$	$-1.18080e-01$	$4.3e-05$	1.4
50.0	>	$1.73901e-01$	$-1.18080e-01$	$2.2e-06$	1.8

Table 1: The master integral $T_{12345}(p^2; m_1^2, m_2^2, m_3^2, m_4^2, m_5^2)$, calculated with the one-dimensional integral representation. The masses are $m_1^2 = 1$, $m_2^2 = 2$, $m_3^2 = 3$, $m_4^2 = 4$, $m_5^2 = 5$. A: '<' indicates, that p^2 is below all thresholds, i.e. the imaginary part is zero. B: Numerical result for the real part. C: Numerical result for the imaginary part. D: Error reported by the QUADPACK-routine. E: CPU-time in seconds.

As expected, the one-dimensional integral representation is considerably faster. The difference is more important if higher accuracy is required. Note that the CPU-time for the one-dimensional integration is nearly the same for the two runs for all values of p^2, while the accuracy has increased by a factor of 20. For the two-dimensional integration, on the other hand, a difference in the computing-time by a factor 5 causes about the same increase in accuracy. While the one-dimensional integration gives results accurate to about 6 digits (regarding the real error) in about 1 second, the two-dimensional integration needs about 85 seconds to achieve an accuracy of between 3 and 4 digits, depending on the parameters.

p^2	A	B	C	D	E
0.1	$<$	$-2.88941e - 01$	$4.46110e - 03$	$3.4e - 03$	16
0.1	$<$	$-2.87005e - 01$	$-2.17356e - 04$	$7.0e - 04$	84
0.5	$<$	$-2.94846e - 01$	$4.17190e - 04$	$1.6e - 03$	16
0.5	$<$	$-2.94795e - 01$	$-4.88930e - 05$	$3.5e - 04$	85
1.0	$<$	$-3.05219e - 01$	$8.17018e - 07$	$1.3e - 03$	17
1.0	$<$	$-3.04930e - 01$	$-3.78437e - 04$	$2.8e - 04$	85
5.0	$<$	$-4.53058e - 01$	$-8.90271e - 04$	$8.3e - 04$	17
5.0	$<$	$-4.52409e - 01$	$-9.65061e - 05$	$1.8e - 04$	86
10.0	$>$	$-4.88270e - 01$	$-3.52664e - 01$	$4.8e - 04$	17
10.0	$>$	$-4.88250e - 01$	$-3.53326e - 01$	$1.1e - 04$	86
50.0	$>$	$1.73864e - 01$	$-1.18080e - 01$	$6.9e - 05$	17
50.0	$>$	$1.73907e - 01$	$-1.18072e - 01$	$1.5e - 05$	85

Table 2: The master integral $T_{12345}(p^2; m_1^2, m_2^2, m_3^2, m_4^2, m_5^2)$, calculated with the two-dimensional integral representation. The masses are $m_1^2 = 1$, $m_2^2 = 2$, $m_3^2 = 3$, $m_4^2 = 4$, $m_5^2 = 5$. A: '$<$' indicates, that p^2 is below all thresholds, i.e. the imaginary part is zero. B: Numerical result for the real part. C: Numerical result for the imaginary part. D: Error reported by the VEGAS-routine. E: CPU-time in seconds.

6   Conclusion

For scalar two-loop self-energy integrals analytic expressions have been presented in the literature for many special cases [5, 6]. Asymptotic expansions and Taylor series are another important approach to these integrals, see e.g. [8]. Furthermore, for diagrams containing a self-energy subloop, series representations have been presented [2]-[4] which are analytical results for the general mass case. No comparable analytical result has so far been found for the master diagram in the general mass case. Anyhow, series representations always have a restricted region of convergence, i.e. in each particular region up to the closest threshold or pseudo-threshold. In the general case of the master diagram with its many parameters this will lead to considerable complications concerning representations in terms of generalized hypergeometric functions or in terms of Taylor series.

In this paper we presented analytical results for the imaginary part of the master diagram T_{12345} and a one-dimensional integral representation for T_{12345}. In all our results the two- and three-particle cut contributions are separated, which gives some additional insight into the analytical structure of the function T_{12345}. Together with corresponding results [3, 4] for the other fundamental two-loop self-energy integrals and with a tensor reduction formalism for the two-loop self-energy integrals [20], efficient algorithms are now available for the calculation of two-loop self-energies in the general mass case. The numerical implementations should be sufficient to perform complete calculations in the standard model at the two-loop level, e.g. to evaluate the fermion or gauge-boson self-energies.

Acknowledgements

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